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Introduction to Circuit QED

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Lecture 1: Introduction to Circuit QED

- ‘Blackbox’ Quantization (BBQ)
- Dispersive Coupling and Readout
- Strong Dispersive Limit
- Photon ‘Number Splitting ‘
- Photon Number Parity Measurement

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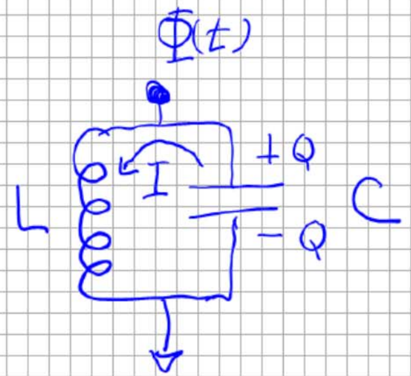
Introduction to Circuit QED

Artificial atoms and microwave photons

How to be a quantum electrical engineer

LC oscillator



Define generalized flux

$$\Phi(t) \equiv \int^{t} d\tau \, V(\tau)$$

$$\dot{\Phi} = V$$

Faraday induction (up to a minus sign)

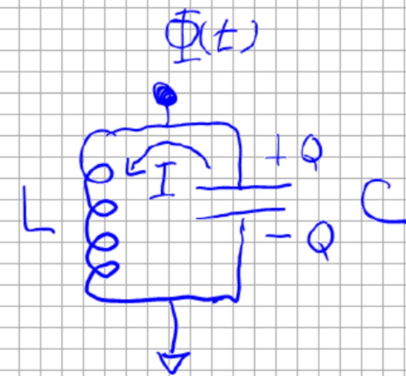
electrostatic energy $\frac{1}{2} C \dot{\Phi}^2$

magnetic energy $\frac{1}{2} L I^2 = \frac{1}{2L} \Phi^2$ $\quad (\Phi = IL)$

Lagrangian $\mathcal{L} = \frac{1}{2} C \dot{\Phi}^2 - \frac{1}{2L} \Phi^2$

$$\mathcal{L} = \frac{1}{2} C \dot{\Phi}^2 - \frac{1}{2L} \Phi^2$$

velocity ↗ (on $\dot{\Phi}$) ↖ coordinate (on Φ)



momentum $Q \equiv \frac{\delta \mathcal{L}}{\delta \dot{\Phi}} = C\dot{\Phi} = CV$

charge Q is momentum canonically conjugate to flux.

Hamiltonian $H = Q\dot{\Phi} - \mathcal{L} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$

harmonic oscillator with "mass" $m = C$

"spring constant" $k = 1/L$

resonance frequency $\omega_R = \sqrt{\frac{k}{m}} = \frac{1}{\sqrt{LC}}$

Hamilton eqn's of motion

$$\dot{\Phi} = \frac{\partial H}{\partial Q} = \frac{Q}{C} = V$$ ✓ Faraday induction

$$\dot{Q} = -\frac{\partial H}{\partial \Phi} = -\frac{\Phi}{L} = -I$$ ✓ charge conservation

$$\ddot{\Phi} = \frac{\dot{Q}}{C} = -\frac{1}{LC}\Phi$$

$$I = I_0 \sin(\omega_R t + \theta)$$

$$V = I_0 Z_R \cos(\omega_R t + \theta)$$

Z_R ← characteristic impedance

$$I = -\dot{Q} = -C\dot{V} = +\underbrace{\omega_R C Z_R}_{=1} I_0 \sin(\omega_R t + \theta)$$

$$\boxed{Z_R = \frac{1}{\omega_R C} = \sqrt{\frac{L}{C}}}$$

$Z_R \sim 50 - 500\,\Omega$ because impedance of free space $Z_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\,\Omega$

quantum of impedance $Z_K \equiv \frac{h}{e^2} \approx 25{,}812\ \Omega$

$$\alpha \equiv \frac{e^2}{\hbar c}\frac{1}{[4\pi\epsilon_0]} \approx \frac{1}{137}$$

$$\boxed{Z_0 = 2\alpha Z_K}$$

Quantizing the oscillator

$$[\hat{Q}, \hat{\Phi}] = -i\hbar \qquad \hat{\Phi} = \Phi_{ZPF}(a + a^{+}) \qquad \hat{Q} = -i\, Q_{ZPF}(a - a^{+})$$

$$[a, a^{+}] = 1 \qquad Q_{ZPF}\,\Phi_{ZPF} = \frac{\hbar}{2}$$

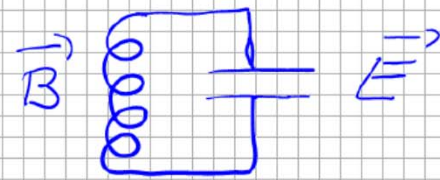
virial thm $\langle 0 | \frac{\hat{Q}^2}{2C} | 0 \rangle = \frac{1}{2}\left(\frac{1}{2}\hbar\omega_R\right) \Rightarrow Q_{ZPF} = \sqrt{\frac{\hbar}{2 Z_R}}$

$$\langle 0 | \frac{\hat{\Phi}^2}{2L} | 0 \rangle = \frac{1}{2}\left(\frac{1}{2}\hbar\omega_R\right) \Longrightarrow \Phi_{ZPF} = \sqrt{\frac{\hbar}{2} Z_R}$$

$Q_{ZPF}\,\Phi_{ZPF} = \frac{\hbar}{2}$ ✓ $\quad \Psi(\Phi) = \langle \Phi | 0 \rangle$ is a minimum uncertainty packet

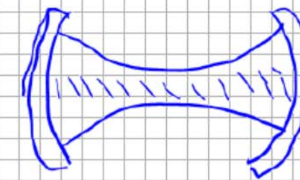
$$\frac{Q_{ZPF}}{e} = \sqrt{\frac{h}{4\pi e^2}\,\frac{1}{Z_R}} = \sqrt{\frac{Z_K}{4\pi Z_R}} \sim \sqrt{\frac{137}{4\pi}} \sim 3$$

$H = \hbar\omega_R (a^\dagger a + \frac{1}{2})$



$a^\dagger$ creates a <u>photon</u>

can also represent 3D free space modes

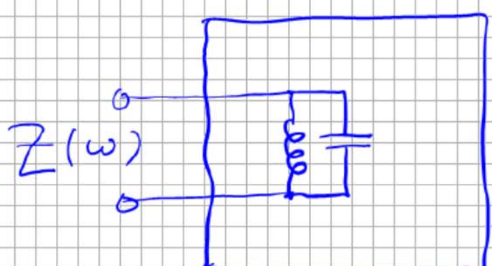


1D optical fibers, co-planar wave guides, etc.

Black Box Quantization (BBQ)

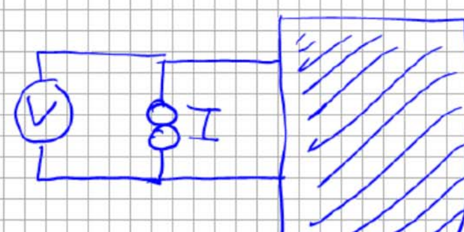
PRL 108, 240502 (2012)

To quantize an oscillator, we only need to know ω_R and Z_R.

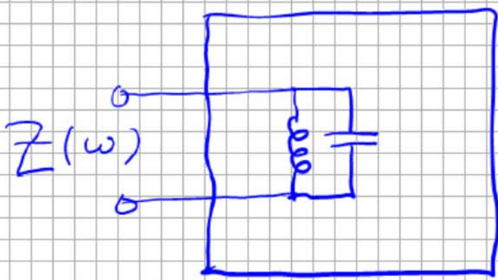


Measure $Z(\omega)$ via $V(\omega) = I(\omega) Z(\omega)$

or compute using a finite-element Maxwell solver e.g. HFSS, Comsol, etc.



Important that ideal current source and voltmeter have <u>infinite</u> impedance.



For a single LC resonator

$$\frac{1}{Z(\omega)} = j\omega C + \frac{1}{j\omega L}$$

[we are engineers! $j = -i$]

$$Z(\omega) = -j\, Z_R \frac{\omega\, \omega_R}{(\omega - \omega_R)(\omega + \omega_R)}$$

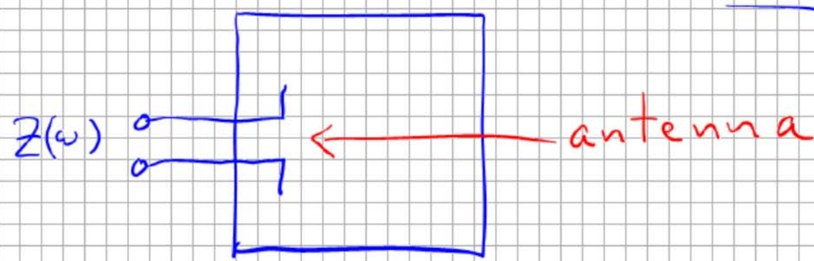
$Z(\omega)$ has poles at $\pm\omega_R$

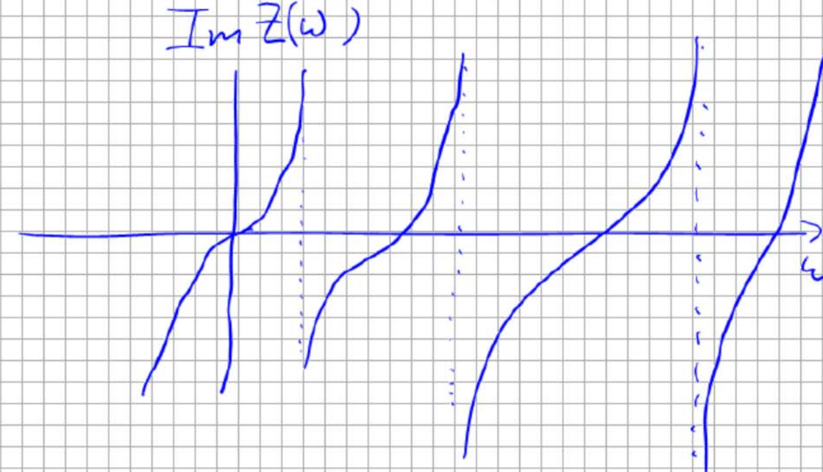
Residue: $\lim_{\omega \to +\omega_R} (\omega - \omega_R) Z(\omega) = -\frac{j Z_R \omega_R}{2}$

Poles of $Z(\omega)$ determine black box resonances under open circuit conditions.

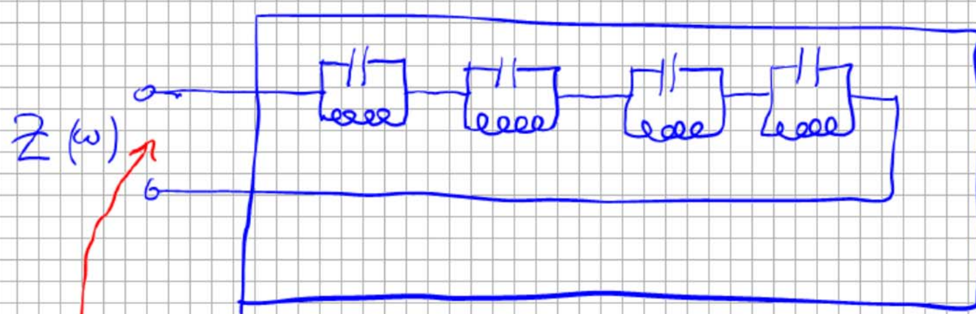
Residues determine characteristic impedances.

General black box has many resonances





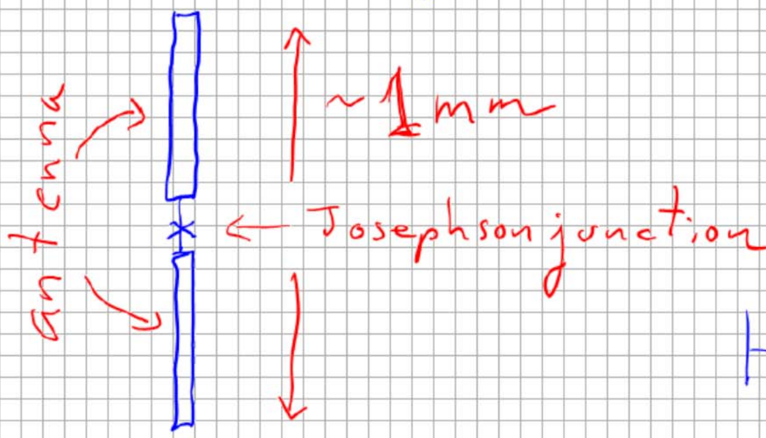
Foster equivalent circuit



$$\lim_{\omega \to \omega_\ell} (\omega - \omega_\ell)\left[\sum_k Z^{(k)}(\omega)\right] = \frac{-j Z_R^{(\ell)} \omega_\ell}{2}$$

$$\hat{\Phi} = \sum_k \Phi_{ZPF}^{(k)} \left(a_k^\dagger + a_k\right)$$

"Transmon" qubit



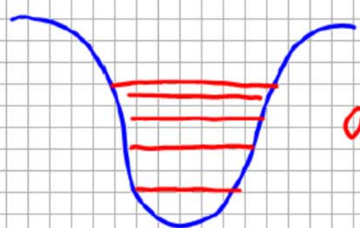
dipole moment

$\sim Q_{ZPF} \times 1\,mm$

$$H = \frac{\hat{Q}^2}{2C_\Sigma} - E_J \cos\Big(\underbrace{\frac{2e}{\hbar}\hat{\Phi}}_{\varphi}\Big)$$

$C_\Sigma \equiv C_J + C_{geometric}$

$\varphi \equiv$ SC order parameter phase



anharmonic oscillator

$\hbar\dot{\varphi} = 2eV = 2e\dot{\Phi}$

Josephson relation

Subtlety: $\hat{Q} = (2e)\,\hat{n}$ is discrete not continuous.

For $E_J \gg E_c \equiv \dfrac{e^2}{2C_\Sigma}$ $\quad \langle \varphi^2 \rangle \ll 2\pi$ so can expand the cosine and safely ignore the subtleties.

Typically $\dfrac{E_J}{E_c} \sim 10^2$

$$H = \frac{\hat{Q}^2}{2C_\Sigma} - E_J \cos\left(\frac{2e}{\hbar}\Phi\right) = H_0 + V$$

$$H_0 = \frac{\hat{Q}^2}{2C_\Sigma} + \underbrace{\frac{E_J}{2}\left(\frac{2e}{\hbar}\right)^2}_{\frac{1}{2L_J}} \Phi^2$$

Josephson plasma oscillation

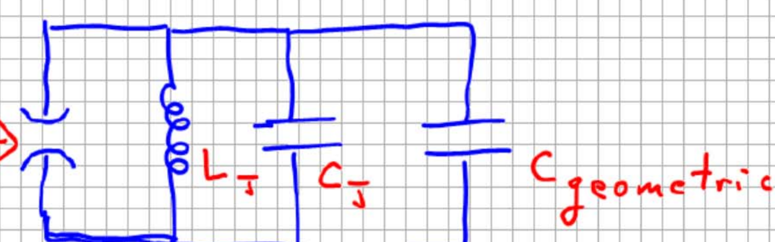
$\frac{1}{2L_J}$ Josephson inductance

$$\hbar\omega_J = \frac{1}{\sqrt{L_J C_\Sigma}} = \sqrt{8E_J E_C} \qquad E_C \equiv \frac{e^2}{2C_\Sigma}$$

$$\frac{\omega_J}{2\pi} \sim 5\text{–}10\ \text{GHz}$$

$$V(\Phi) = E_J\left\{-\frac{1}{4!}\left(\frac{2e}{\hbar}\Phi\right)^4 + \frac{1}{6!}\left(\frac{2e}{\hbar}\Phi\right)^6 - \cdots\cdots\right\}$$

Expand cosine, remove quadratic part



$$V \simeq -\frac{E_C}{2}\left(a^\dagger a^\dagger a a + 2a^\dagger a\right)$$

(rotating wave approximation)

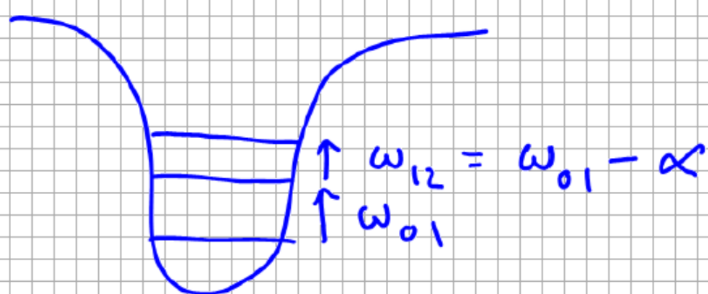
$$H = \hbar \omega_{01}\, a^\dagger a - \frac{\hbar\alpha}{2} a^\dagger a^\dagger a a$$

$$\hbar\omega_{01} = \hbar\omega_J - E_c$$

$$\hbar\alpha = E_c$$

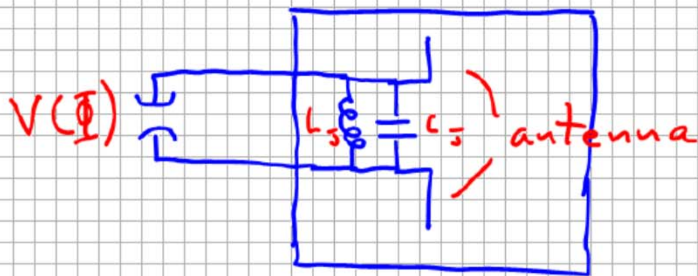
$\alpha \sim 200\ \mathrm{MHz}$ $\omega_{01} \sim 5\ \mathrm{GHz}$

'weakly' anharmonic oscillator



For Rabi driving at ω_{01} with amplitude $\Omega_R \ll \alpha$ anharmonicity looks <u>large</u> and transmon $\simeq$ 2-level atom

Transmon coupled to resonator (BBQ)



$$\hat{\Phi} = \sum_k \Phi_{ZPF}^{(k)} \left(a_k + a_k^\dagger\right)$$

Diagonalize quadratic H_0 ; Express V in that basis

$$V = -\sum_\ell \delta\omega_\ell A_\ell^\dagger A_\ell + \sum_{jk} \chi_{jk} \hat{n}_j \hat{n}_k$$

Most anharmonic mode identified as qubit (say $j=0$)

χ_{00} big χ_{0k} next biggest $\chi_{\ell k}$ smallest $\ell, k \neq 0$

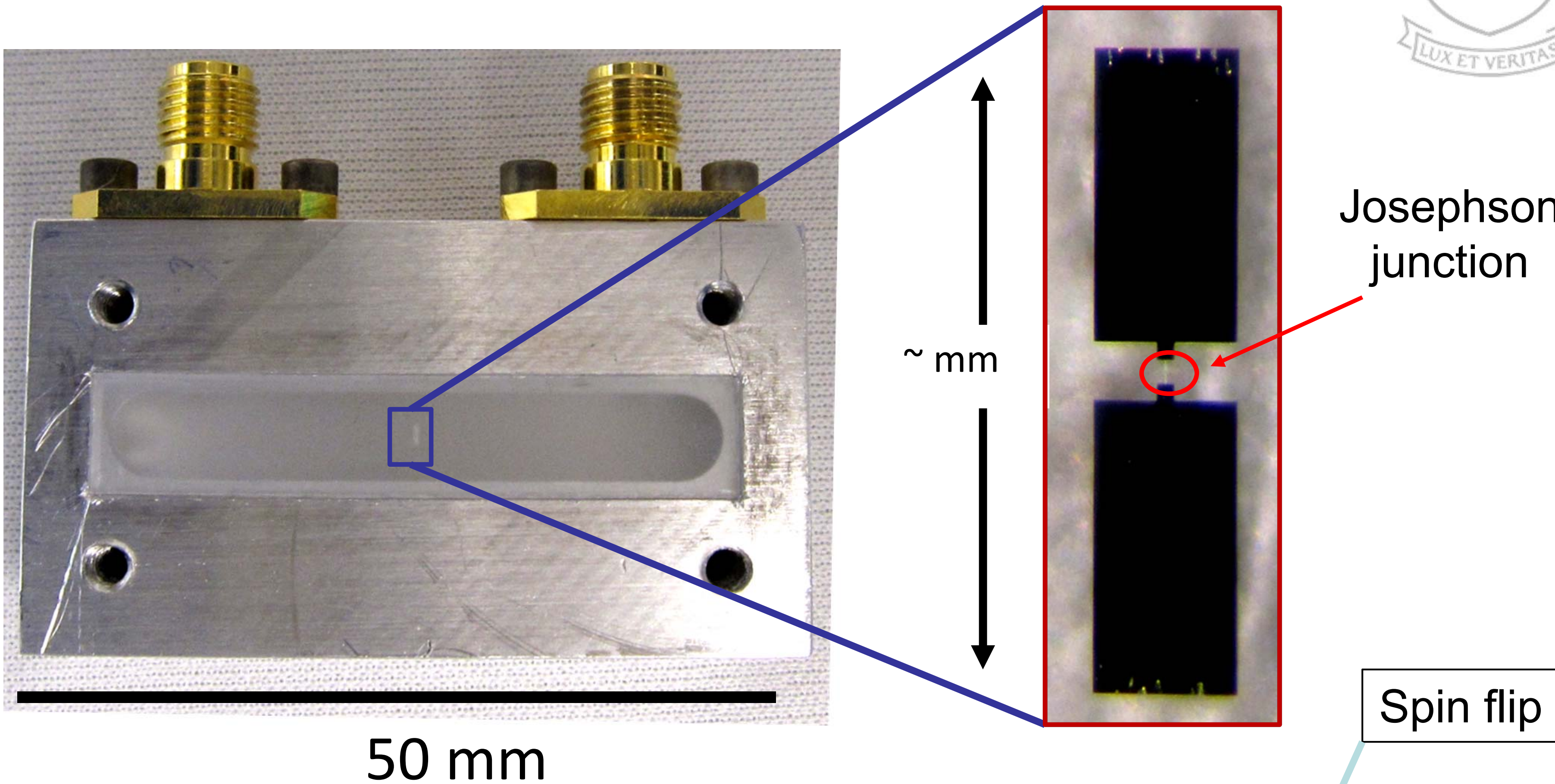
$\hat{n}_0 = 0, 1 \longleftrightarrow \sigma^z = \pm 1$ two-level approximation

$$\chi_{0k} \hat{n}_0 \hat{n}_k \longrightarrow \frac{\chi_{0k}}{2} \sigma^z \hat{n}_k$$

cross Kerr ⟶ Jaynes-Cummings in dispersive limit

mode frequency depends on qubit state

Transmon Qubit in 3D Cavity



$$g = \frac{\vec{d} \bullet \vec{E}_{rms}}{h}$$

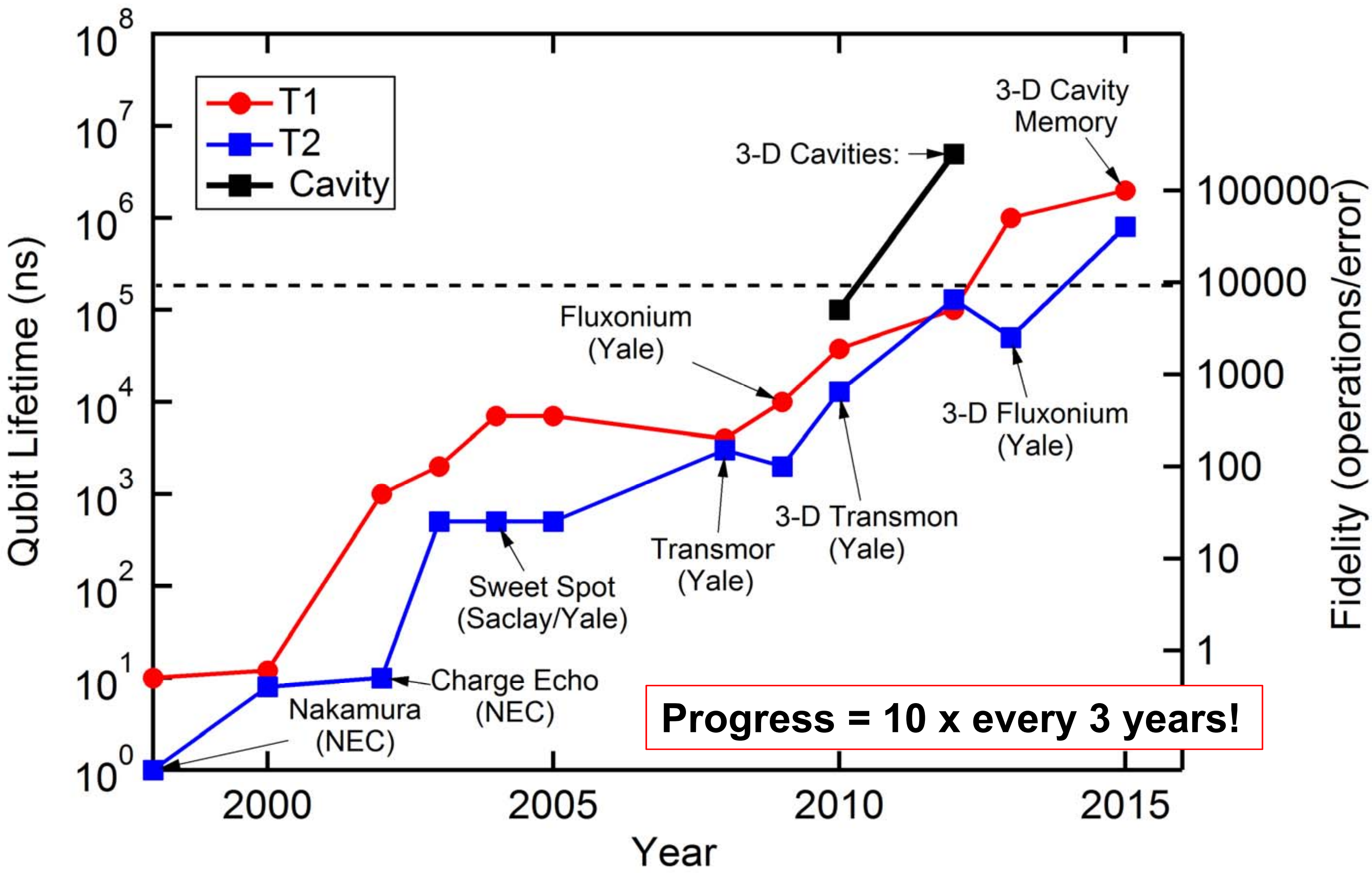
$$\left|\vec{d}\right| = 2e \times 1 \text{ mm} \approx 10^7 \text{Debye!!}$$

Huge dipole moment: strong coupling

$$V_{\text{dipole}} = g\sigma^x(a + a^\dagger)$$

g ~ 100 MHz

Remarkable Progress in Coherence



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Transmission Lines and Input-Output Theory

equivalent circuit:

ℓ = inductance/length

c = capacitance/length

characteristic impedance $Z_0 \equiv \sqrt{\ell/c} \sim 50\,\Omega$

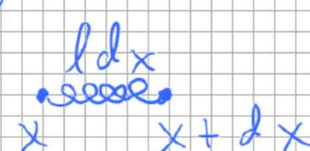
wave velocity $v_0 = \frac{1}{\sqrt{\ell c}}$ (frequency x length)

flux $\Phi(x,t)$

voltage $V(x,t) = \partial_t \Phi(x,t)$

charge density $\rho(x,t) = c\,\partial_t \Phi(x,t)$

current $i(x,t) = -\frac{1}{\ell}\partial_x \Phi(x,t)$ (inductance ℓdx)



$$\rho(x,t) = c\,\partial_t \Phi$$

$$i(x,t) = -\frac{1}{\ell}\,\partial_x \Phi$$

charge conservation $\Rightarrow$ continuity eqn.

$$\partial_t \rho + \partial_x i = 0$$

$$c\,\partial_t^2 \Phi - \frac{1}{\ell}\,\partial_x^2 \Phi = 0$$

$$\partial_t^2 \Phi - v_0^2\,\partial_x^2 \Phi = 0$$

wave equation

cf. Bosonization

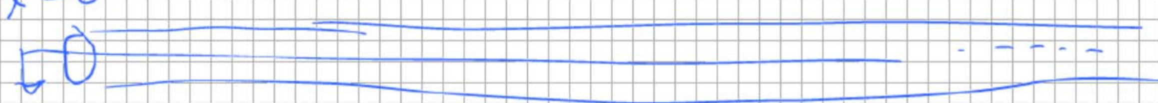
solution:

$$\Phi(x,t) = \Phi_R\left(t - \frac{x}{v_0}\right) + \Phi_L\left(t + \frac{x}{v_0}\right)$$

arbitrary functions!

short circuit bc. $\partial_t \Phi(0,t) = 0$

$x=0$



$$\Phi(0,t) = \Phi_R(t) + \Phi_L(t) = 0$$

$$\Phi_R(t) = \Phi_{out} = -\Phi_{in} = -\Phi_L$$

reflection coefficient $r(\omega) = -1$

open boundary condition

$$i(0,t) = 0 \implies \partial_x \Phi(x,t) = 0$$

$$\partial_x \Phi_R\left(t - \frac{x}{v}\right)\Big|_{x=0} = -\frac{1}{v}\,\partial_t \Phi_R(t)$$

$$\partial_x \Phi_L\left(t + \frac{x}{v}\right)\Big|_{x=0} = +\frac{1}{v}\,\partial_t \Phi_L(t)$$

$$\Phi_{out} = \Phi_R = \Phi_{in} = \Phi_L \qquad r(\omega) = +1$$

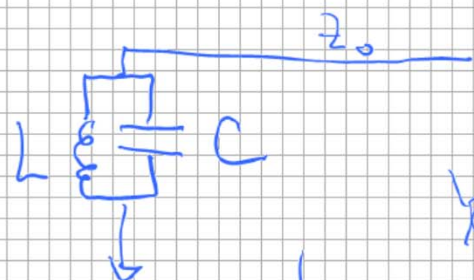
General (linear) case

$$r = \frac{Z(\omega) - Z_0}{Z(\omega) + Z_0}$$

$Z(\omega) = 0 \Rightarrow r = -1$ ✓

$Z(\omega) = \infty \Rightarrow r = +1$

$Z(\omega) = Z_0 \Rightarrow r = 0$

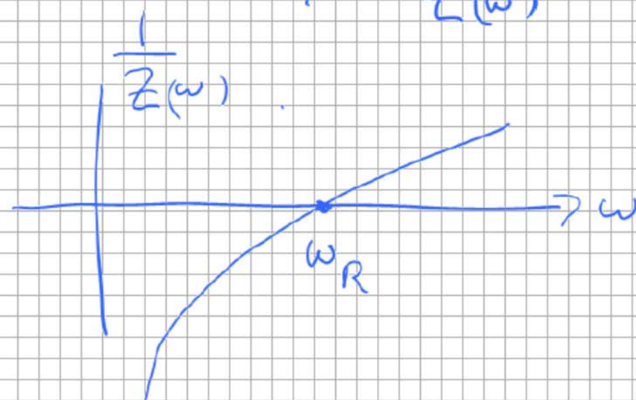


$$Y(\omega) = \frac{1}{Z(\omega)} = j\omega C + \frac{1}{j\omega L}$$

engineering notation

$$\left(j = -\sqrt{-1} \right)$$

$e^{j\omega t}$



$$\omega_R \equiv \frac{1}{\sqrt{LC}}$$

$$Z_R \equiv \sqrt{\frac{L}{C}}$$

$Y(\omega_R) = 0 \quad Y'(\omega_R) = jC - \frac{C}{j} = 2jC$

$$Y(\omega) \simeq 2jC(\omega - \omega_R)$$

$$r(\omega) = \frac{Z(\omega) - Z_0}{Z(\omega) + Z_0} \simeq \frac{1 - 2jCZ_0(\omega - \omega_R)}{1 + 2jCZ_0(\omega - \omega_R)}$$

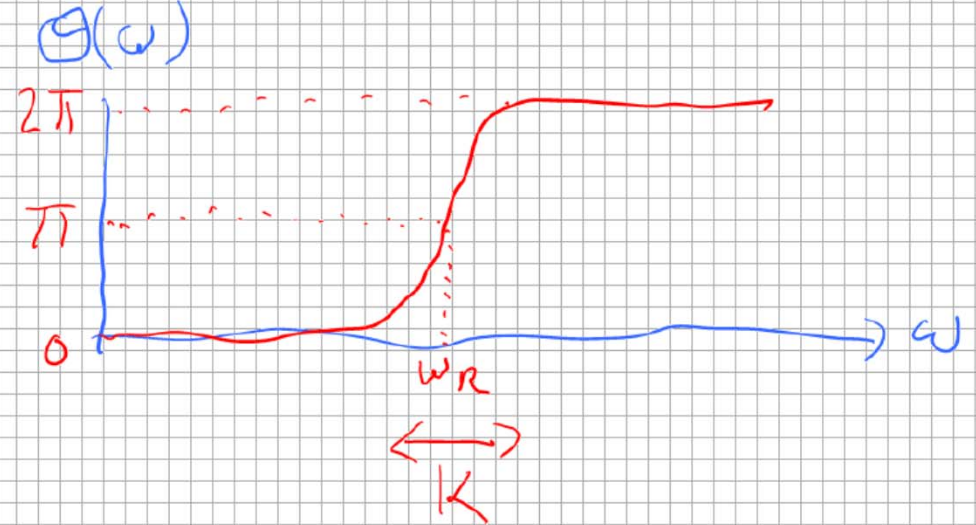
$$\tau = \kappa^{-1} = CZ_0$$

$$r(\omega) \simeq \frac{\omega - \omega_R + j\kappa/2}{\omega - \omega_R - j\kappa/2}$$

$$|r| = 1$$

no internal damping of osc.

$$r(\omega) = e^{i\theta(\omega)}$$



Qubit-Cavity cross-Kerr for two lowest levels of dressed transmon.

$$\chi A^\dagger A B^\dagger B$$

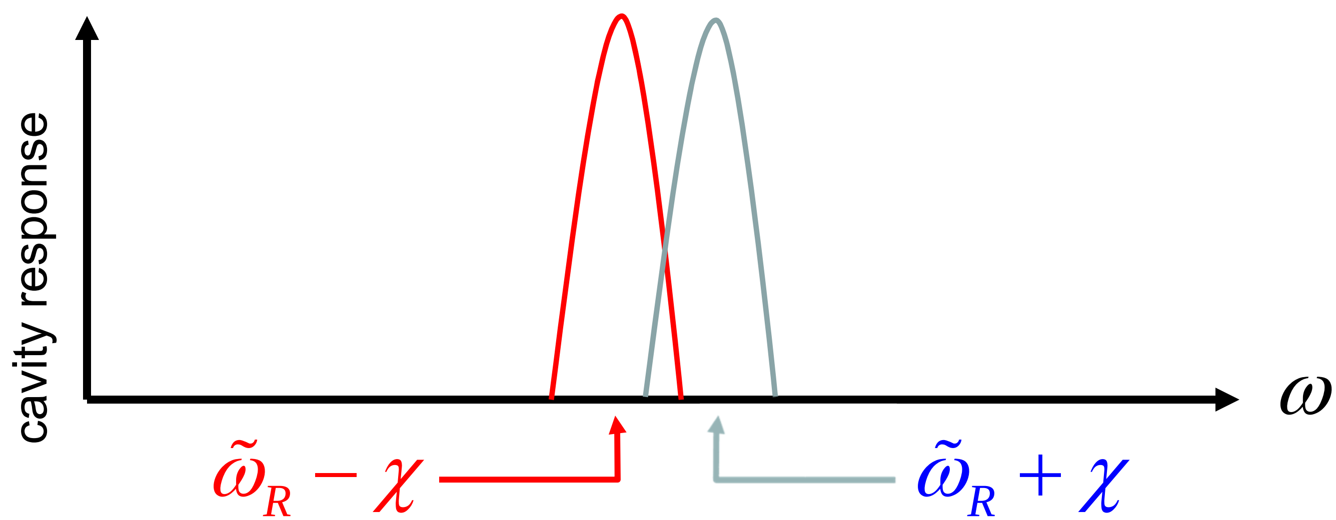
$$A^\dagger A = \hat{n}$$

$$B^\dagger B = \frac{1+\sigma^Z}{2} = 0,1$$

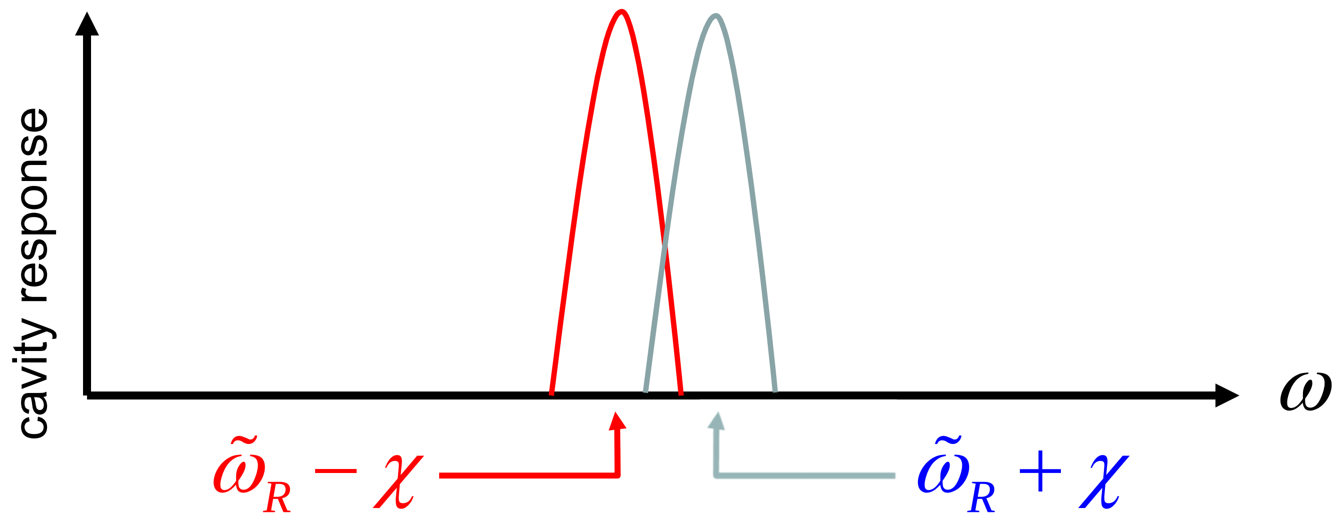
$$H = \left(\tilde{\omega}_R + \chi\sigma^Z\right)\hat{n} + \frac{\omega_Q}{2}\sigma^Z$$

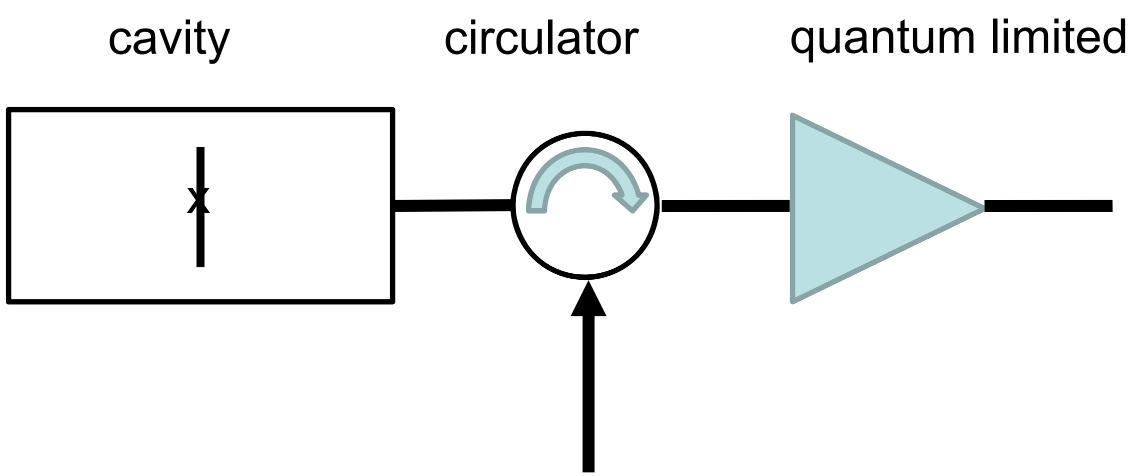
'Dispersive' coupling

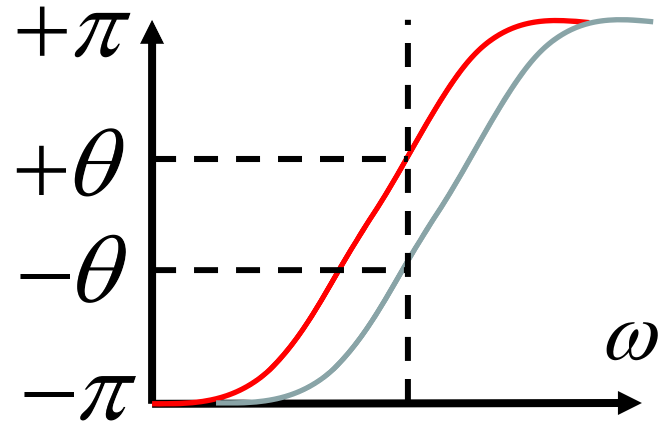
Can read out qubit state by measuring cavity resonance frequency

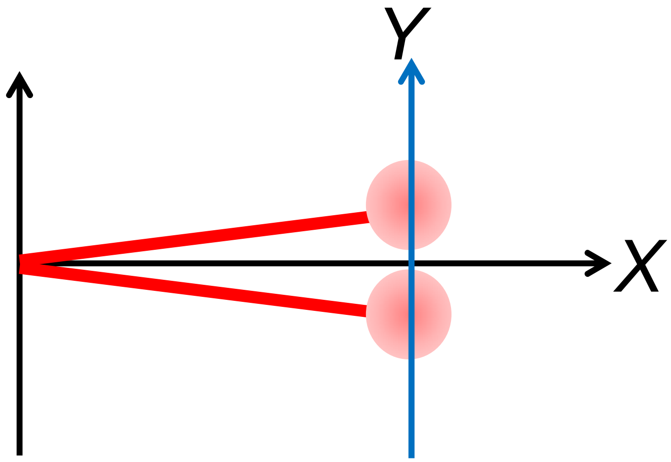


Can read out qubit state by measuring cavity resonance frequency





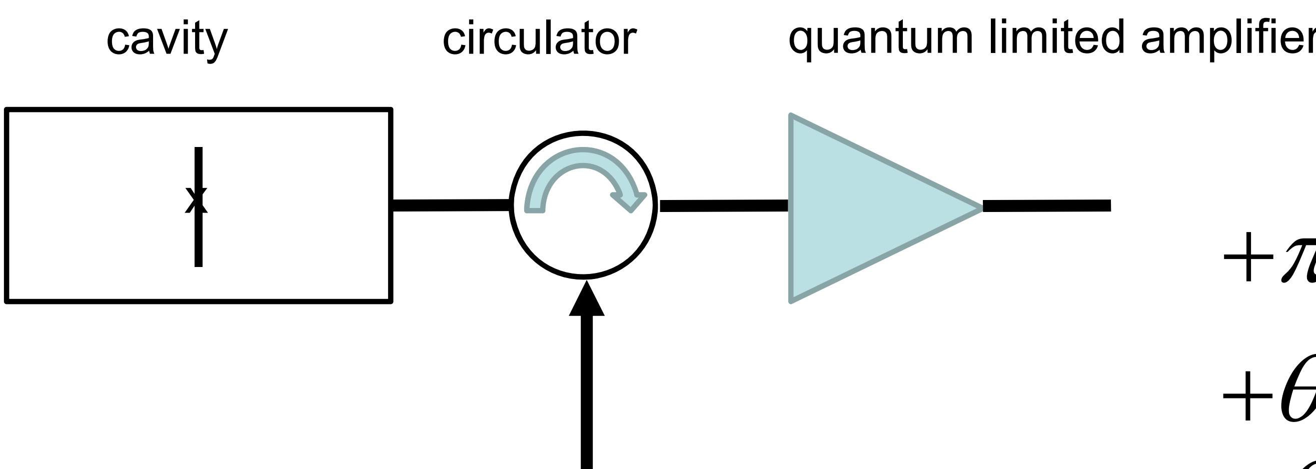


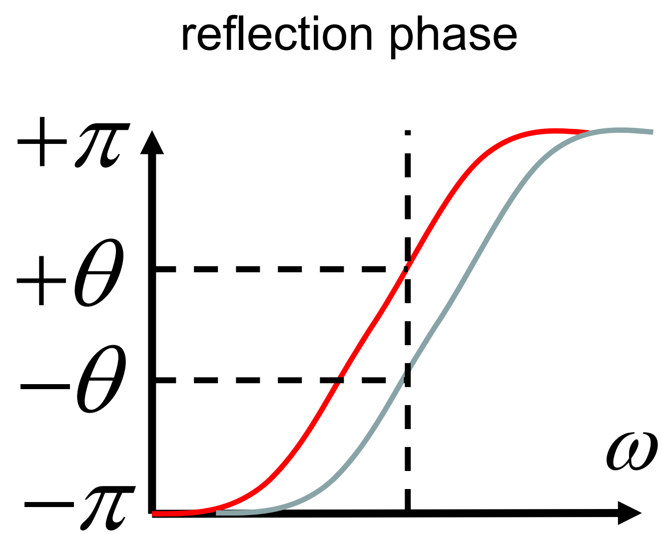


$$|\psi_{\text{in}}\rangle = \{a|\uparrow\rangle + b|\downarrow\rangle\}|\alpha\rangle$$

$$|\psi_{\text{out}}\rangle = a|e^{+i\theta}\alpha\rangle|\uparrow\rangle + b|e^{-i\theta}\alpha\rangle|\downarrow\rangle$$

State of qubit is <u>entangled</u> with the 'meter' (microwave phase)
Then 'meter' is read with amplifier.

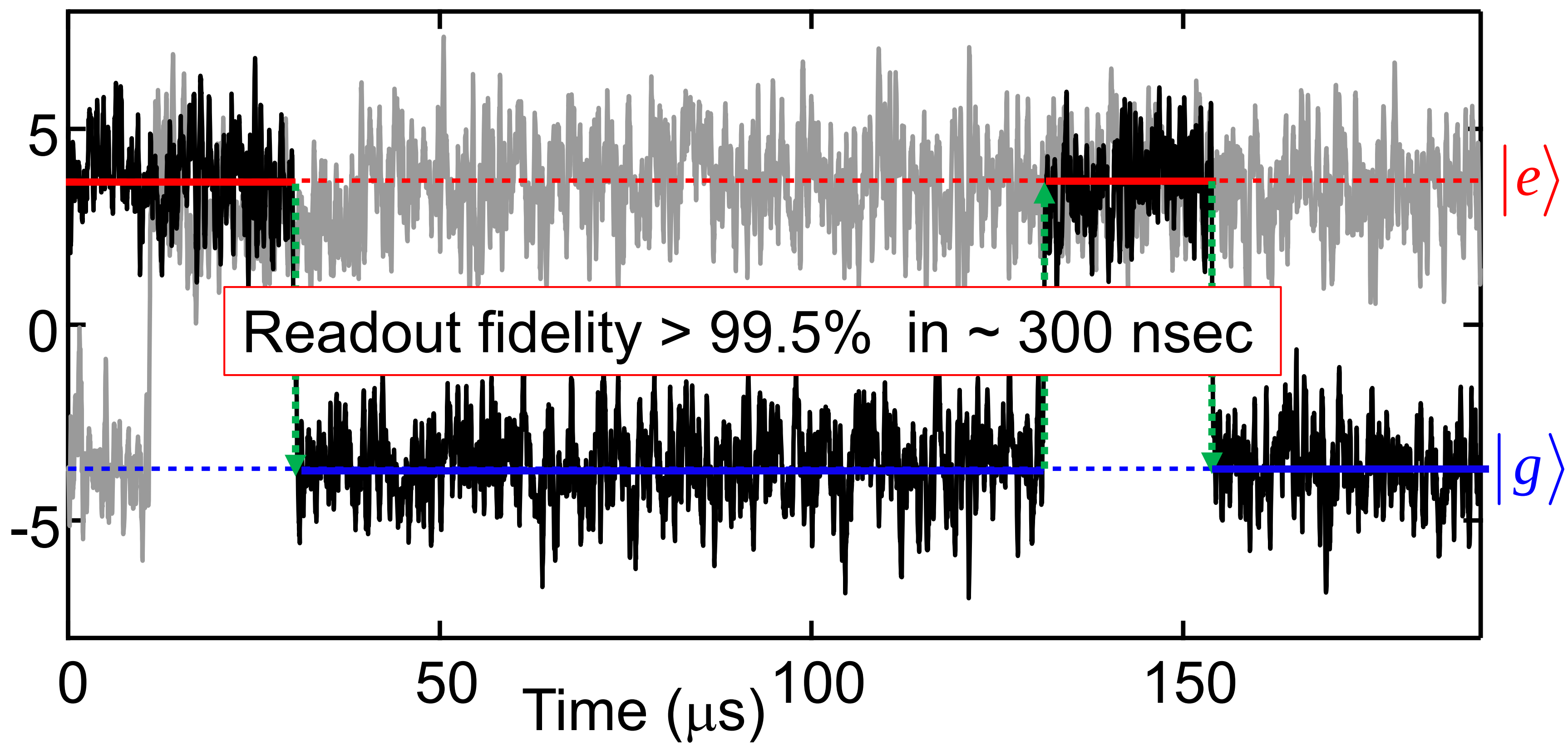




Quantum Jumps of a 3D Transmon Qubit

Results from Devoret group, Yale: Hatridge et al., Science 2013*

dispersive circuit QED readout + JJ paramp



Many groups now working with JJ paramps & feedback, including:
Berkeley, Delft, JILA, ENS/Paris, IBM, Wisc., Saclay, UCSB, …

*First jumps: R. Vijay et al., 2011 (Berkeley)

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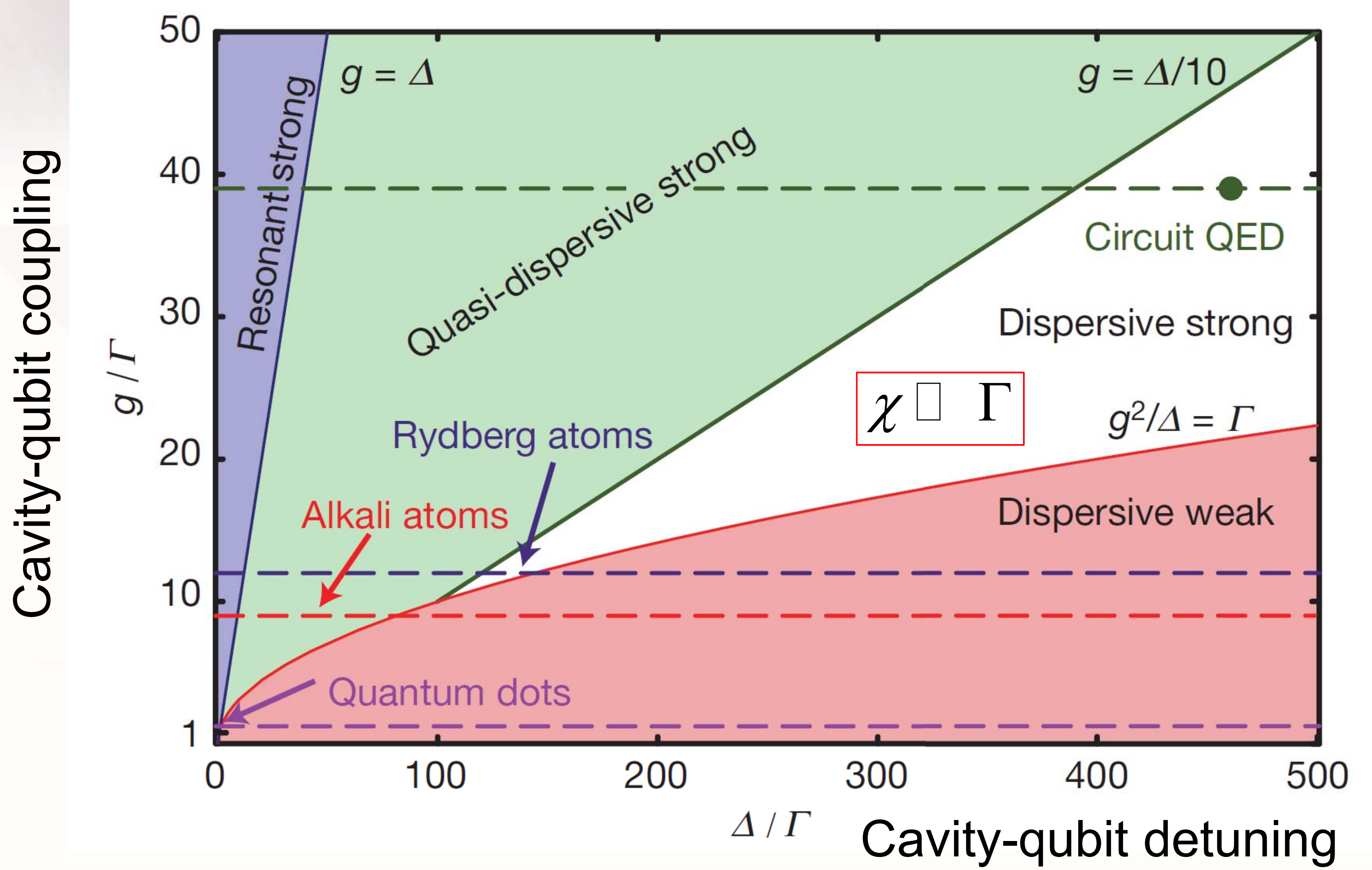
cQED 'phase diagram'

$\Gamma \equiv (\kappa, \gamma)$, linewidth of cavity or qubit



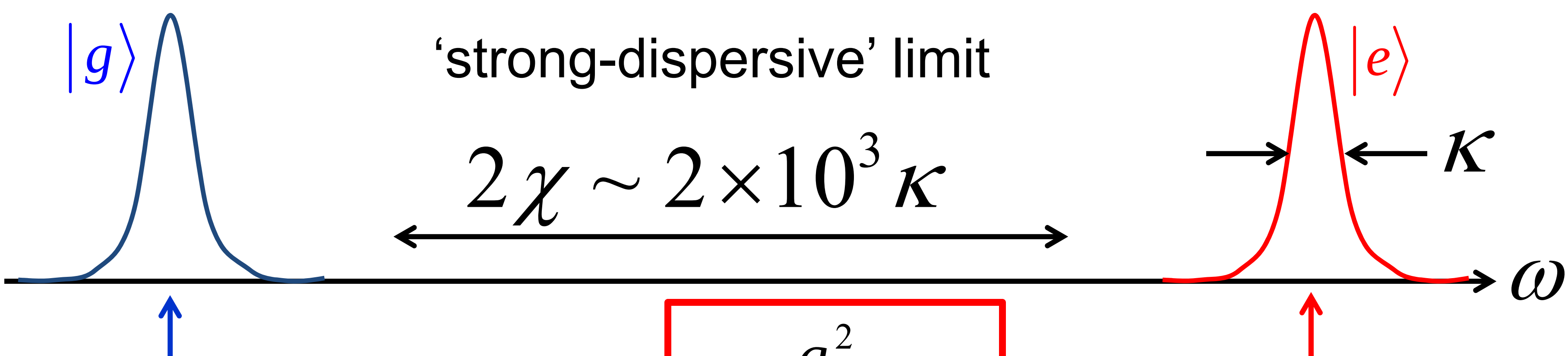
Strong Dispersive Hamiltonian

$$H = \omega_{\mathrm{r}} a^{\dagger} a + \frac{\omega_{\mathrm{q}}}{2} \sigma^{z} + \chi \sigma^{z} a^{\dagger} a + H_{\mathrm{damping}}$$

resonator qubit dispersive coupling

$$\chi >> \kappa, \Gamma$$

$$\text{cavity frequency} = \omega_{\mathrm{r}} + \chi \sigma^{z}$$



$\omega_{\mathrm{r}} - \chi$

$$\chi \sim \frac{g^{2}}{\Delta} \gg \kappa$$

$\omega_{\mathrm{r}} + \chi$

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Using strong-dispersive coupling to measure the photon number distribution in a cavity

Strong Dispersive Hamiltonian

$$H = \omega_{\mathrm{r}} a^\dagger a + \frac{\omega_{\mathrm{q}}}{2}\sigma^z + \chi\sigma^z a^\dagger a + H_{\mathrm{damping}}$$

$$\chi >> \kappa, \Gamma$$

resonator qubit dispersive coupling

Quantized Light Shift of Qubit Transition Frequency

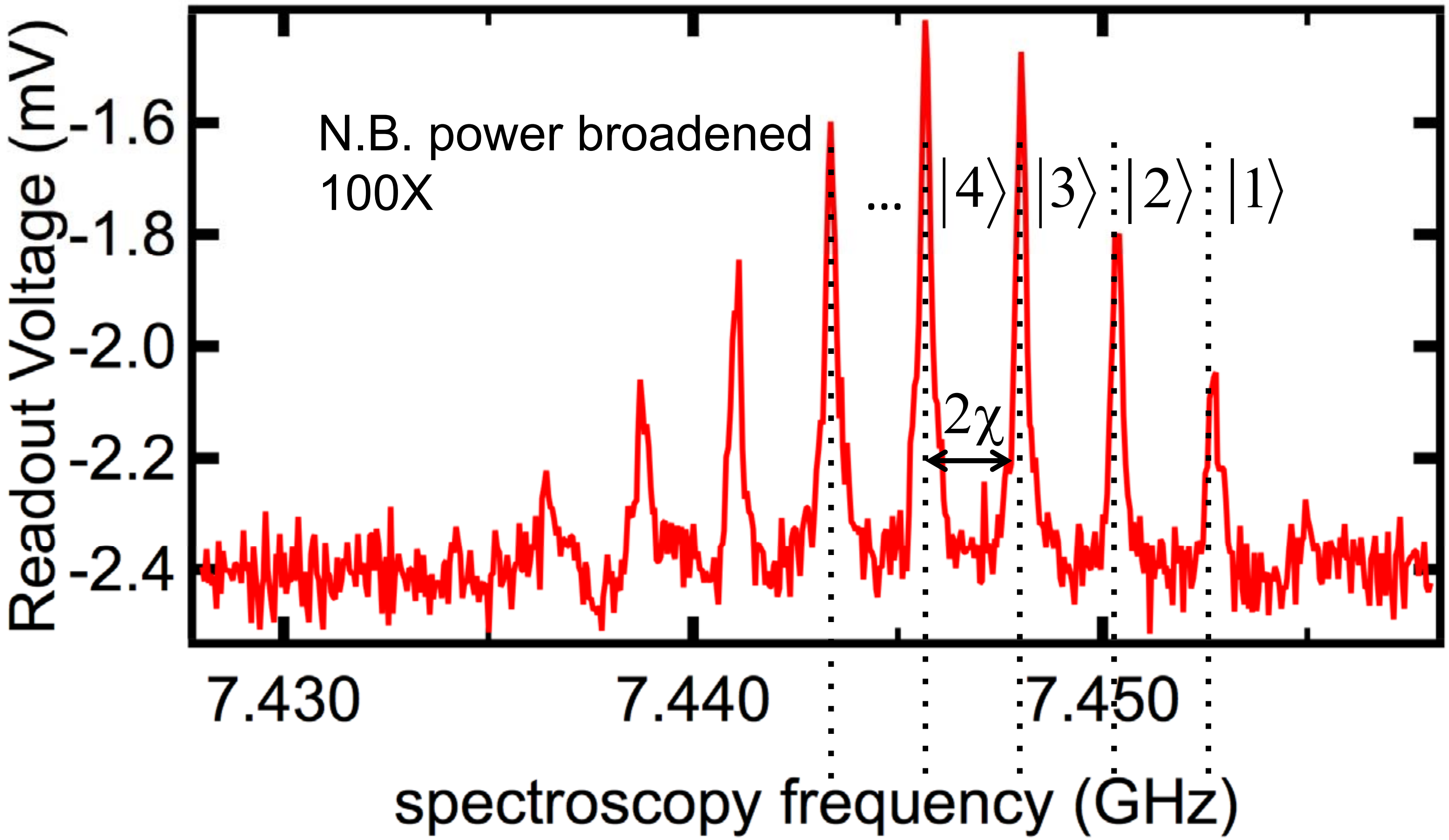
$$H = \omega_{\mathrm{r}} a^\dagger a + \frac{1}{2}\sigma^z \left[\omega_{\mathrm{q}} + 2\chi a^\dagger a\right] + H_{\mathrm{damping}}$$

Coherent state is closest thing to a classical sinusoidal RF signal

$$\psi(\Phi) = \psi_0(\Phi - \alpha)$$

- quantized light shift of qubit frequency (coherent microwave state)

$$\frac{\omega_q + 2\chi a^\dagger a}{2}\sigma^z$$



New low-noise way to do axion dark matter detection?
Zhang et al. arXiv:1607.02529

Lecture 1: Introduction to Circuit QED

- 'Blackbox' Quantization (BBQ)
- Dispersive Coupling and Readout
- Strong Dispersive Limit
- Photon 'Number Splitting '
- Photon Number Parity Measurement

Photon number parity

$$\hat{P} = (-1)^{a^\dagger a} = \sum_{n=0}^{\infty} |n\rangle (-1)^n \langle n|$$

Remarkably easy to measure using our quantum engineering toolbox

and

Measurement is 99.8% QND

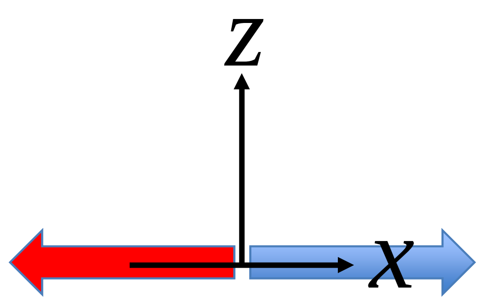
Measuring Photon Number Parity

- use quantized light shift of qubit frequency

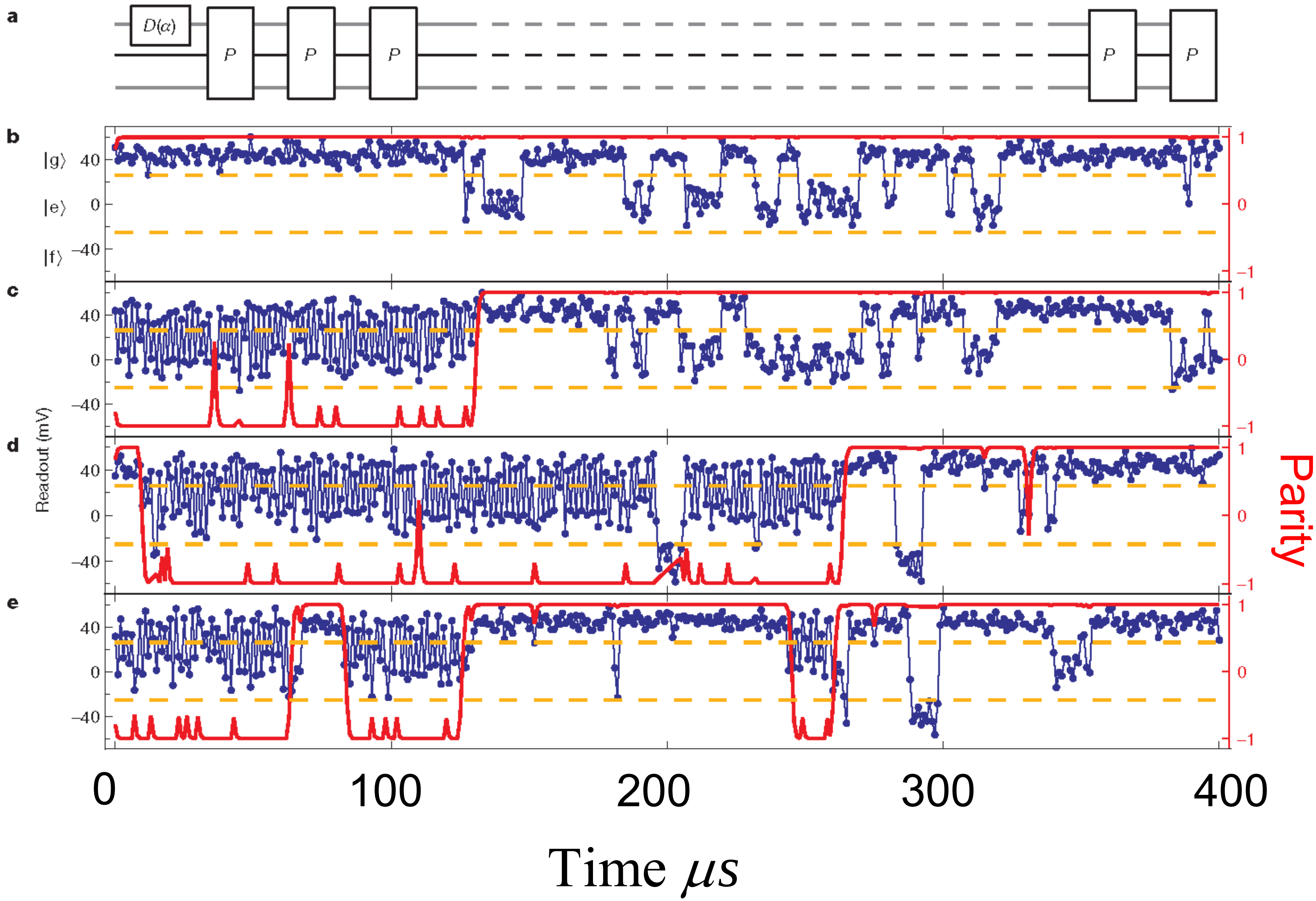
$$\frac{\omega_{\mathrm{q}} + 2\chi a^{\dagger} a}{2}\sigma^{z}$$

$$e^{-i2\chi\hat{n}t\frac{\sigma^{z}}{2}} = \mathrm{e}^{-i\pi\hat{n}\frac{\sigma^{z}}{2}}$$

$\hat{n} = 1, 3, 5, \ldots$



$\hat{n} = 0, 2, 4, \ldots$



Nature **511**, 444 (2014)

400 consecutive parity measurements (99.8% QND)

The ability to measure photon number parity <u>without</u> measuring photon number is an incredibly powerful tool.

<u>Lecture 2</u>: Using parity measurements for:

- Wigner Function Measurements
- Creation and verification of photon cat states

<u>Lecture 3</u>: Using parity measurements for:

- Continuous variable quantum error correction