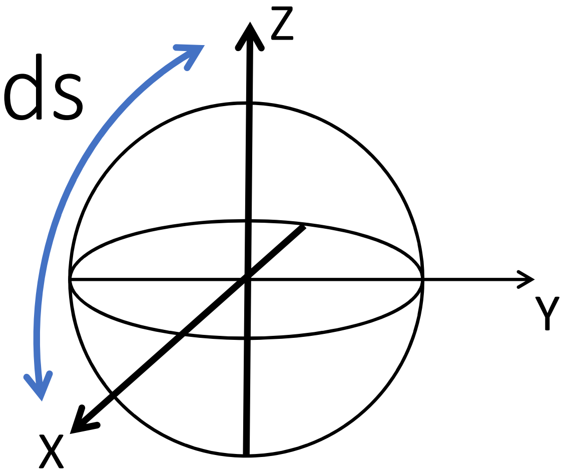
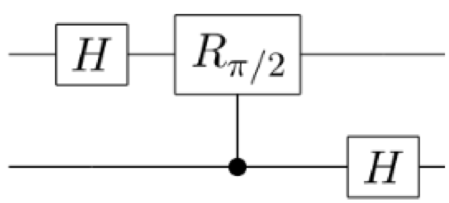
QFT fun on the IBM QX

Building QFT Circuits from Hadamards



- 1-qubit Hadamard **exchanges *Z* and *X* axes**
  - *Z* and *X* are subject to an uncertainty relation like time and frequency → ***Fourier transform!***
- *m*-qubit Hadamard is equivalent to Hadamards on each individual qubit
  - Can be recursively defined, with $H_0 = 1$ and $H_m = \frac{1}{\sqrt{2}}\begin{pmatrix} H_{m-1} & H_{m-1} \\ H_{m-1} & -H_{m-1} \end{pmatrix}$
- *m*-qubit QFT is constructed from Hadamard gates and controlled phase gates
- Example: 2-qubit QFT in matrix and circuit form (verify they are equivalent!)

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \end{pmatrix}$$



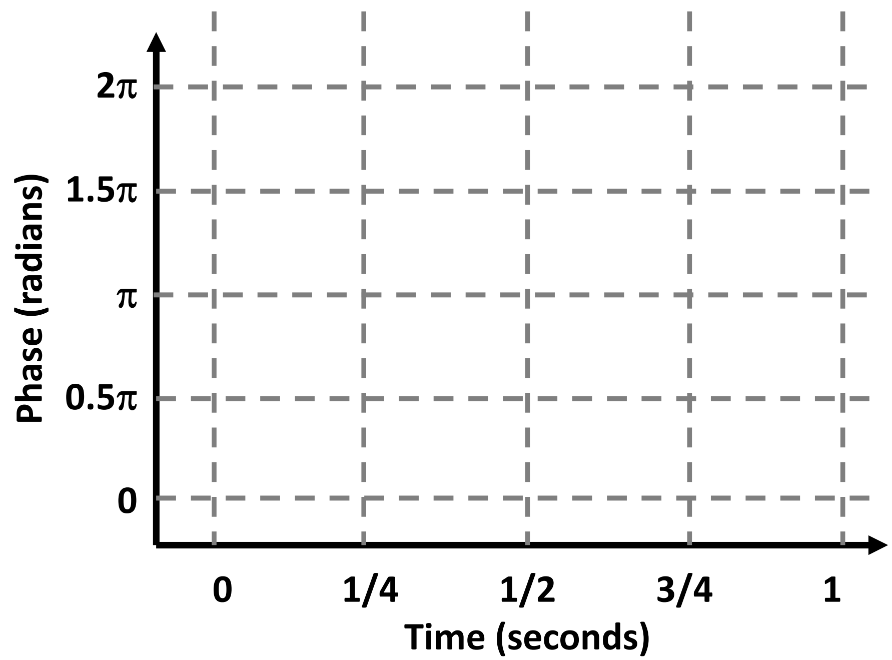
where $R_\theta = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$

More on the QFT

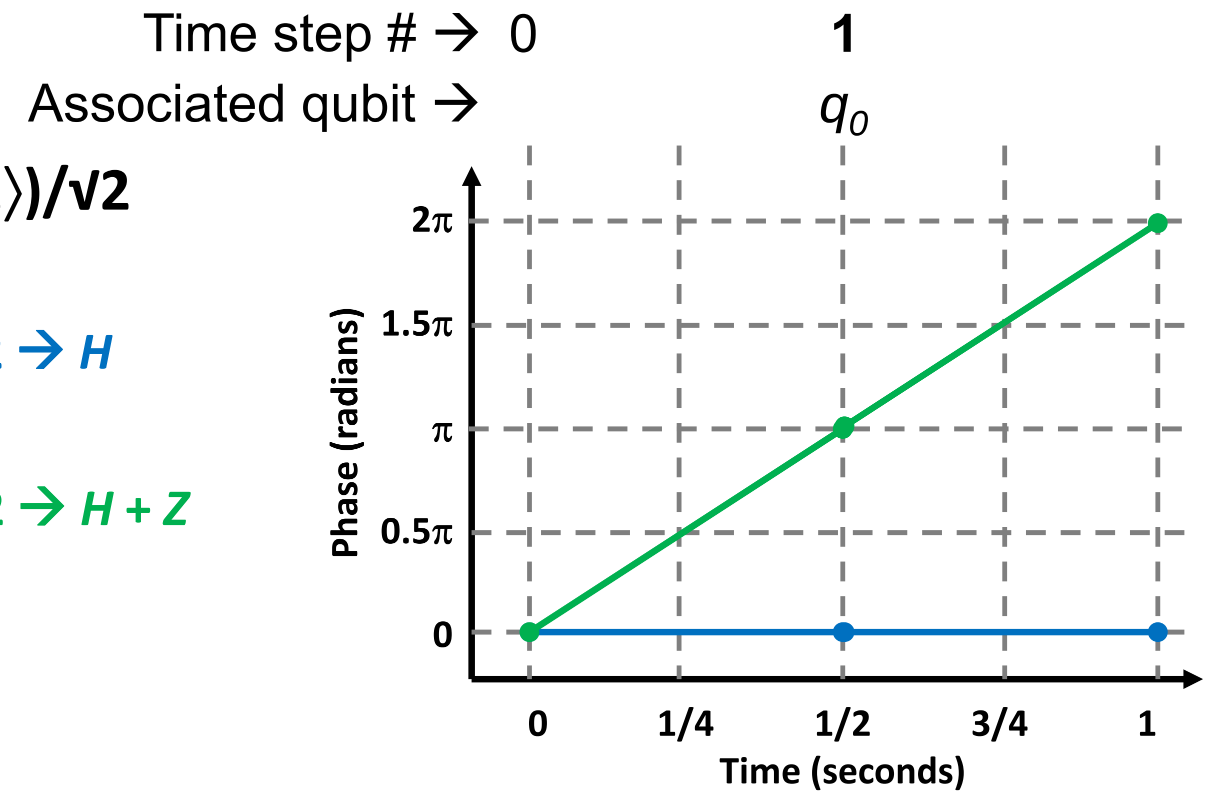
- QFT is a key building block for many quantum algorithms
  - Period finding, phase estimation, factoring, …
- Actual signal processing is **not** among these!
  - QFT performs the Fourier transform of a quantum state itself, not of a signal represented by qubits in $|0\rangle$ and $|1\rangle$ states
- Nonetheless, useful intuition may be gained from a toy example in which we encode various "signals" in the states of a set of qubits

Encoding a signal to be decoded by QFT

- Imagine a periodic function on a time interval from 0 to 1 seconds
  - At time *t*, function has phase $\varphi(t)=2\pi ft$
- Given an *m*-qubit register $|q_m \ldots q_0\rangle$, divide the interval into 2^m steps
- Encode function's phase at time $t = 1/2^{(m-n)}$ s in the relative phase between |0> and |1> on qubit q_n:
  $|q_n\rangle = (|0\rangle + e^{2\pi ift}|1\rangle)/\sqrt{2}$
  → **Phase represented by position on equator of Bloch sphere**
- Preparing qubit register $|q_m \ldots q_0\rangle$ in this way and then applying *m*-qubit Hadamard puts answer bit *i* on qubit *m-i* (→ read bits backwards)



Physical intuition for the QFT: 1 qubit



$t = 1/2^{(1-n)}$: $|q_n\rangle = (|0\rangle + e^{2\pi i f t}|1\rangle)/\sqrt{2}$

- **f = 0 Hz:**
  - **$t = 1/2$: $|q_0\rangle = (|0\rangle + e^{0i}|1\rangle)/\sqrt{2}$ → H**
- **f = 1 Hz:**
  - **$t = 1/2$: $|q_0\rangle = (|0\rangle + e^{\pi i}|1\rangle)/\sqrt{2}$ → $H + Z$**

QX demo: m-qubit QFT, $m = 1$

For each frequency $f < 2^m$ Hz:

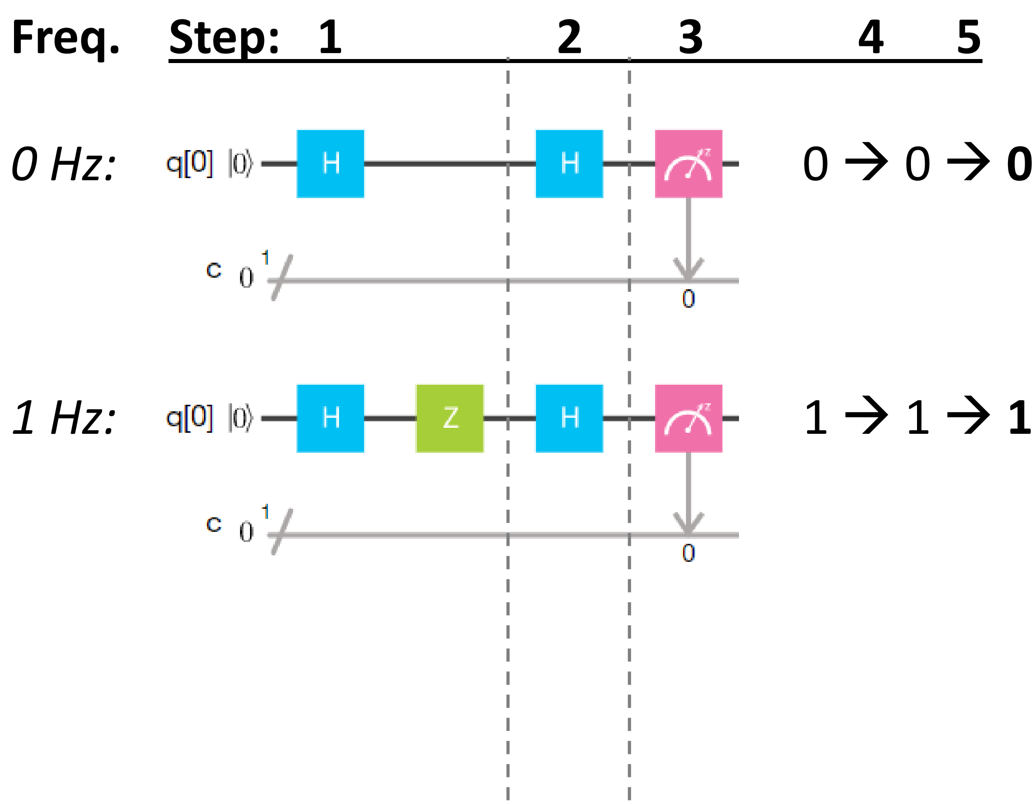
1. Prepare each qubit q_n in $(|0\rangle + e^{2\pi i f t}|1\rangle)/\sqrt{2}$, $t = 1/2^{(m-n)}$

2. Perform QFT

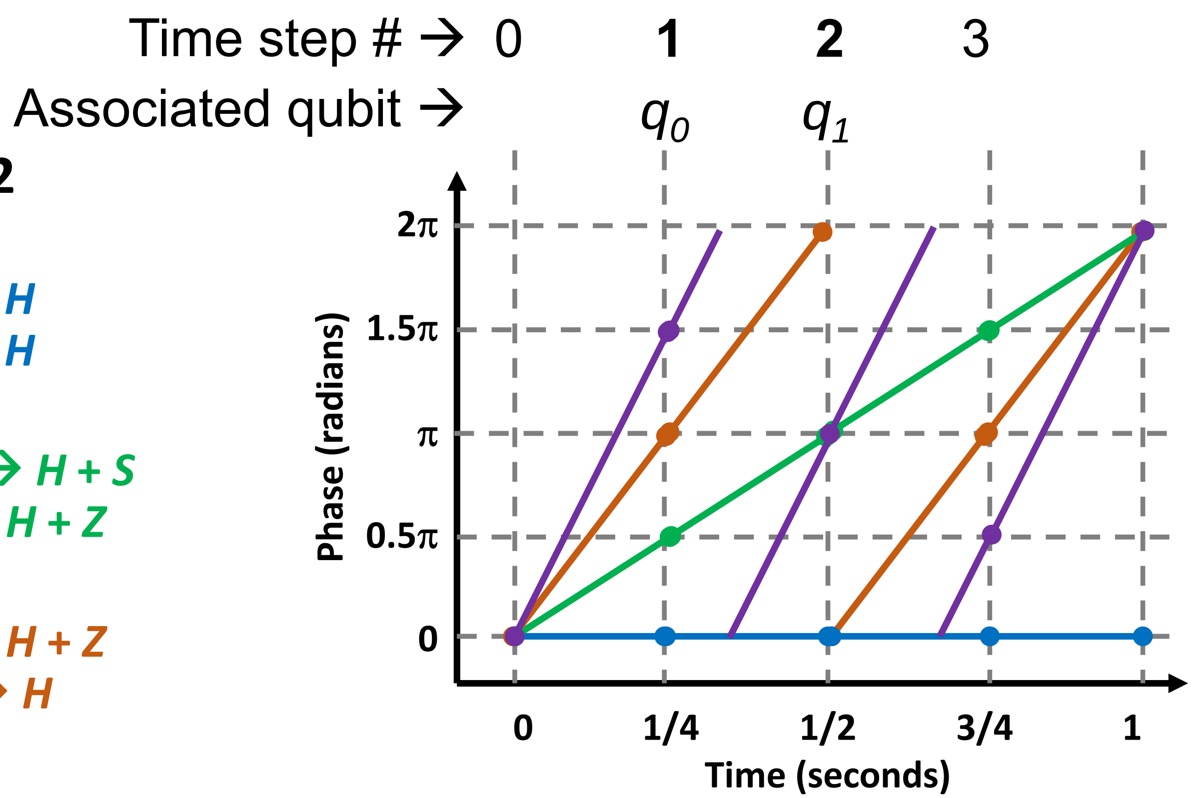
3. Measure all q_n

4. Reverse the bit order (in principle this could be done with a series of SWAP gates prior to measurement)

5. Convert binary to decimal; should recover f



Physical intuition for the QFT: 2 qubits



$t = 1/2^{(2-n)}$: $|q_n\rangle = (|0\rangle + e^{2\pi i f t}|1\rangle)/\sqrt{2}$

- **f = 0 Hz:**
  - $t = 1/4$: $|q_0\rangle = (|0\rangle + e^{0i}|1\rangle)/\sqrt{2} \rightarrow H$
  - $t = 1/2$: $|q_1\rangle = (|0\rangle + e^{0i}|1\rangle)/\sqrt{2} \rightarrow H$
- **f = 1 Hz:**
  - $t = 1/4$: $|q_0\rangle = (|0\rangle + e^{i\pi/2}|1\rangle)/\sqrt{2} \rightarrow H + S$
  - $t = 1/2$: $|q_1\rangle = (|0\rangle + e^{\pi i}|1\rangle)/\sqrt{2} \rightarrow H + Z$
- **f = 2 Hz:**
  - $t = 1/4$: $|q_0\rangle = (|0\rangle + e^{\pi i}|1\rangle)/\sqrt{2} \rightarrow H + Z$
  - $t = 1/2$: $|q_1\rangle = (|0\rangle + e^{2\pi i}|1\rangle)/\sqrt{2} \rightarrow H$
- **f = 3 Hz: (*left as an exercise*)**

QX demo: m-qubit QFT, $m = 2$

For each frequency $f < 2^m$ Hz:

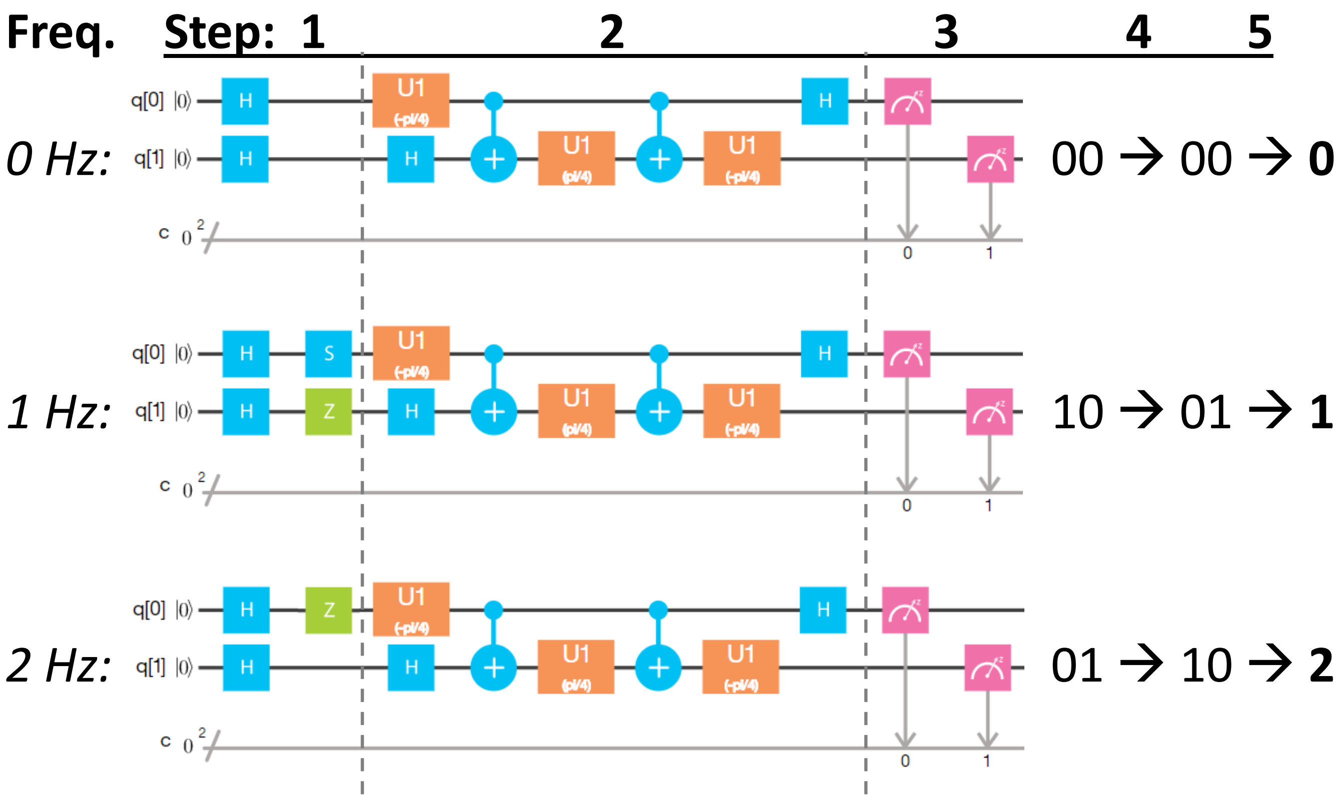
1. Prepare each qubit q_n in $(|0\rangle + e^{2\pi i f t}|1\rangle)/\sqrt{2}$, $t = 1/2^{(m-n)}$

2. Perform QFT

3. Measure all q_n

4. Reverse the bit order (in principle this could be done with a series of SWAP gates prior to measurement)

5. Convert binary to decimal; should recover f

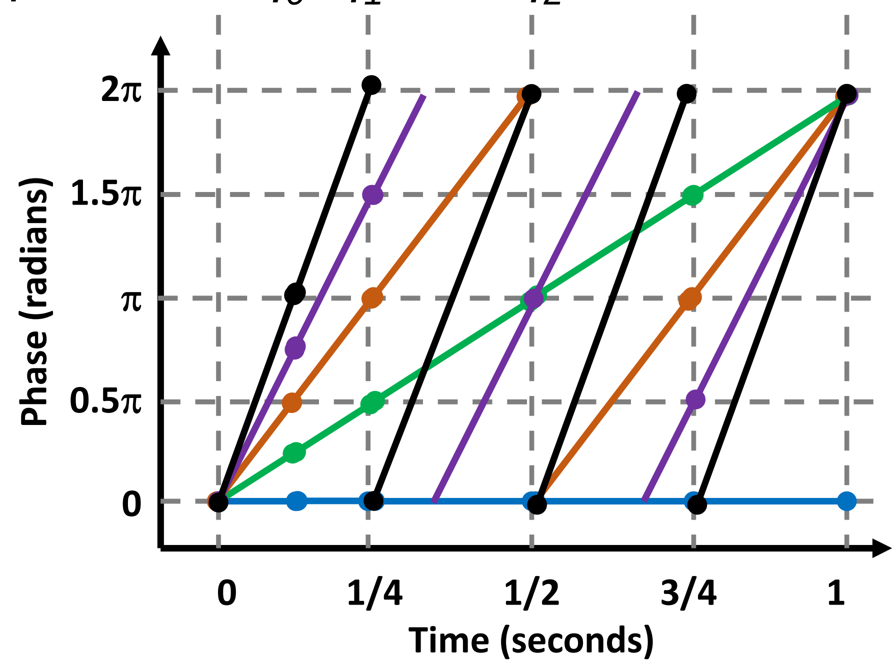


3 Hz: (left as an exercise)

Physical intuition for the QFT: 3 qubits

Time step # → 0 **1** **2** 3 **4** 5 6 7

Associated qubit → q_0 q_1 q_2



$t = 1/2^{(3-n)}$: $|q_n\rangle = (|0\rangle + e^{2\pi i f t}|1\rangle)/\sqrt{2}$

- **f = 0 Hz: (*still trivial: H on all 3 qubits*)**
- **f = 1 Hz:**
  - $t = 1/8$: $|q_0\rangle = (|0\rangle + e^{i\pi/4}|1\rangle)/\sqrt{2} \rightarrow H + T$
  - $t = 1/4$: $|q_1\rangle = (|0\rangle + e^{i\pi/2}|1\rangle)/\sqrt{2} \rightarrow H + S$
  - $t = 1/2$: $|q_2\rangle = (|0\rangle + e^{\pi i}|1\rangle)/\sqrt{2} \rightarrow H + Z$
- **f = 2 Hz:**
  - $t = 1/8$: $|q_0\rangle = (|0\rangle + e^{i\pi/2}|1\rangle)/\sqrt{2} \rightarrow H + S$
  - $t = 1/4$: $|q_1\rangle = (|0\rangle + e^{\pi i}|1\rangle)/\sqrt{2} \rightarrow H + Z$
  - $t = 1/2$: $|q_2\rangle = (|0\rangle + e^{2\pi i}|1\rangle)/\sqrt{2} \rightarrow H$
- **f = 3 Hz:**
  - $t = 1/8$: $|q_0\rangle = (|0\rangle + e^{3\pi i/4}|1\rangle)/\sqrt{2} \rightarrow H + S + T$
  - $t = 1/4$: $|q_1\rangle = (|0\rangle + e^{3\pi i/2}|1\rangle)/\sqrt{2} \rightarrow H + S^{\dagger}$
  - $t = 1/2$: $|q_2\rangle = (|0\rangle + e^{\pi i}|1\rangle)/\sqrt{2} \rightarrow H + Z$
- **f > 3 Hz: (*left as an exercise*)**

QX demo: m-qubit QFT, m = 3

For each frequency $f < 2^m$ Hz:

1. Prepare each qubit q_n in $(|0\rangle + e^{2\pi i f t}|1\rangle)/\sqrt{2}$, $t = 1/2^{(m-n)}$

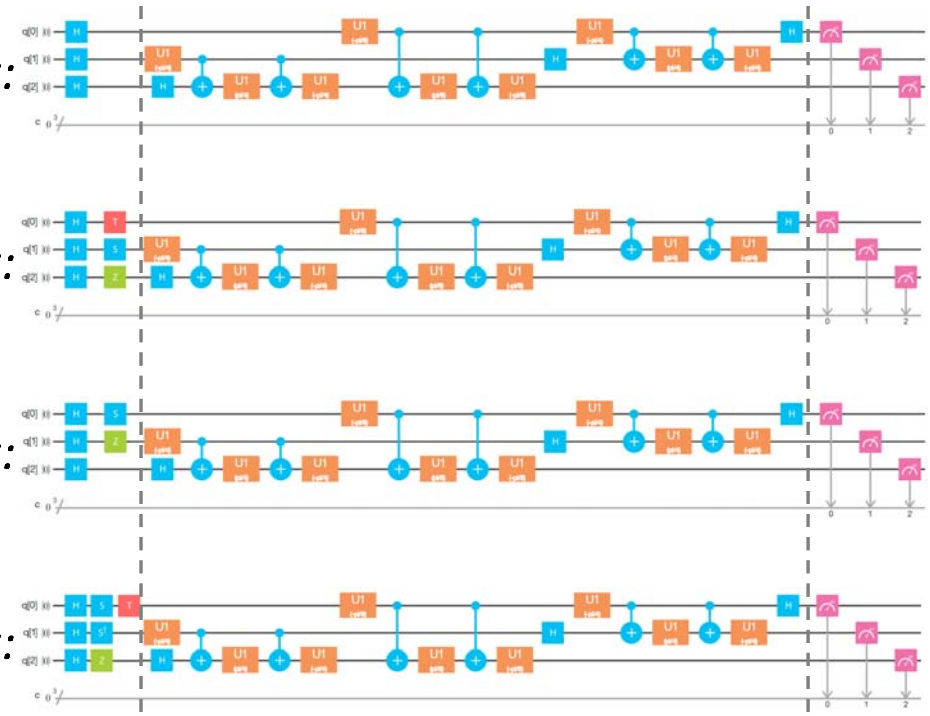
2. Perform QFT

3. Measure all q_n

4. Reverse the bit order (in principle this could be done with a series of SWAP gates prior to measurement)

5. Convert binary to decimal; should recover f

Freq. 1 2 3 4 5



0 Hz: 000 → 000 → **0**

1 Hz: 100 → 001 → **1**

2 Hz: 010 → 010 → **2**

3 Hz: 110 → 011 → **3**

> 3 Hz: (left as an exercise)

QFT Summary

- Single qubit: Hadamard equivalent to quantum Fourier transform
- Many qubits: QFT composed of Hadamards and controlled-phase gates
- QFT transforms quantum states, not signals
- Enables a measurement in the computational basis to reveal information about phase relationships
- Next step: see how the QFT figures into important quantum algorithms like period finding and phase estimation